

Isotopies of high genus Lagrangian surfaces

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1 Introduction

In this paper we address a local version of the isotopy problem for Lagrangian surfaces in a symplectic 4-manifold (M, ω) . This question was first raised by V. Arnold in [1]. A Lagrangian submanifold L is one for which $\omega|_L$ vanishes. In general we would like to classify homotopic Lagrangian submanifolds up to smooth isotopy or better still Lagrangian isotopy, that is, smooth isotopy through Lagrangian submanifolds. Equivalence classes are called Lagrangian knots. Here we show that in a sufficiently small neighborhood of a given Lagrangian surface there are no Lagrangian knots up to smooth isotopy. More precisely our result can be stated as follows.

Theorem 1.1. *Let $T^*\Sigma$ be the cotangent bundle of a Riemann surface with its canonical symplectic structure and $L \subset T^*\Sigma$ be a connected Lagrangian submanifold homologous to Σ . Then L is smoothly isotopic to Σ .*

In the case when Σ has genus 0 or 1 the above Theorem 1.1 is due to Y. Eliashberg and L. Polterovich, see [6]. In fact, work of the first author, see [13], shows that if Σ has genus 0 then all such Lagrangian spheres are Lagrangian isotopic to the zero-section. However we remark that it is not true in general that isotopic spheres are Lagrangian isotopic, see the work of P. Siedel, [26]. If Σ has genus 1 work of the second author [20] shows again that all such L are Lagrangian isotopic. The question of whether or not in higher genus cases all such isotopic Lagrangians are Lagrangian isotopic remains open.

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In general symplectic 4-manifolds there exist homologous high genus Lagrangian submanifolds which are not smoothly isotopic, see the work of D. Park, M. Poddar and S. Vidussi, [24].

2 Proof of the theorem

In this section we prove Theorem 1.1. Since the result is known when Σ has genus 0 or 1 we will assume throughout that Σ has genus $g > 1$. Let σ be an area form on Σ of total area $2g - 2$. Let $\pi : T^*\Sigma \rightarrow \Sigma$ be the projection along the fibers. The cotangent bundle $T^*\Sigma$ carries a canonical symplectic form $\omega_0 = d(\lambda_0)$, where $\lambda_0 = p d\pi$ is the Liouville form. The zero section Σ is Lagrangian with respect to ω_0 .

We can also think of $T^*\Sigma$ as a tubular neighborhood of a symplectic submanifold Σ . Then $T^*\Sigma$ carries another symplectic form τ which is symplectic on the fibers and such that $\tau|_\Sigma = \sigma$. Let $r : T^*\Sigma \rightarrow [0, \infty)$ be the length function with respect to an Hermitian metric on (the complex line bundle) $T^*\Sigma$. We denote the levels by $T^r\Sigma$. Then the unit circle bundle $\pi : T^1\Sigma \rightarrow \Sigma$ carries a connection α with $d\alpha = \pi^*\sigma$. We can arrange that $\tau|_{T^r\Sigma} = f(r)d\tilde{\alpha}$ where $\tilde{\alpha}$ is the pullback of the form α on $T^1\Sigma$ and f is decreasing towards 0 as r approaches ∞ .

For ϵ sufficiently small, $\Omega_\epsilon = \omega_0 + \epsilon\tau$ is also a symplectic form on $T^*\Sigma$.

We reparameterize ω_0 such that outside of a large compact set $T^{\leq r_0}\Sigma$ it is given by $d(e^r \tilde{\lambda}_0)$ where $\tilde{\lambda}_0$ denotes the pullback of the Liouville form from the unit tangent bundle. Also outside of $T^{\leq r_0}\Sigma$ we extend τ by extending the function f to a decreasing function $g(r)$ with $g = -e^r$ outside of a (larger) compact set. Then we define a new form ω on $T^*\Sigma$ by $\omega = \Omega_\epsilon$ on $T^{\leq r_0}\Sigma$ and $\omega = d(e^r \tilde{\lambda}_0 + \epsilon g(r) \tilde{\alpha})$ elsewhere. We note that ω is a symplectic form for ϵ sufficiently small and that the fibers of $T^*\Sigma$ are ω -symplectic planes of infinite area.

Let V be a tubular neighborhood of our Lagrangian submanifold $L \subset (T^{\leq r_0}\Sigma, \omega_0)$.

Lemma 2.1. *There exists an $\epsilon_0 > 0$ such that for all $\epsilon < \epsilon_0$ the Lagrangian L can be isotoped to an Ω_ϵ symplectic surface within V .*

Proof This is a slight modification of Proposition 2.1.A in [6]. Let σ be a symplectic form on V such that $\sigma|_L$ is an area form of total area $2g - 2$. Then $(\tau - \sigma)|_L$ is exact and so by the relative Poincaré Lemma there exists

a 1-form λ on V such that $\sigma = \tau + d\lambda$. Let $\rho : V \rightarrow [0, 1]$ have compact support and equal 1 close to L . Then there exists an ϵ_0 such that for $\epsilon \leq \epsilon_0$ the form $\Omega'_\epsilon = \omega_0 + \epsilon(\tau + d(\rho\lambda))$ is symplectic as is the linear family of forms connecting Ω'_ϵ to Ω_ϵ . As $\Omega'_\epsilon = \Omega_\epsilon$ away from V and L is Ω'_ϵ symplectic it follows from Moser's method that L can be isotoped to an Ω_ϵ symplectic surface inside V . \square

The results from [14] imply that that all Ω_ϵ symplectic surfaces sufficiently close to Σ are isotopic to Σ . Thus we could conclude here if it were possible to arrange that the symplectic surface was contained in a suitably small symplectic neighborhood. However we could find no straightforward method of doing this. Instead we proceed as follows.

Lemma 2.2. *For ϵ sufficiently small, all connected symplectic surfaces S in $(T^*\Sigma, \omega)$ which are homologous to Σ and intersect the fiber over a point p exactly once transversally must be smoothly isotopic to Σ .*

Proof Let U be a neighborhood of p such that a given symplectic surface S intersects all fibers over points $q \in U$ transversally in a single point. By a small perturbation we may assume that $S \cap \Sigma$ is disjoint from $\pi^{-1}(U)$.

Let $h : \Sigma \rightarrow \mathbb{R}$ be a Morse function with a single minimum and all critical points contained in U . Then the gradient flowlines of h foliate the complement of the critical points of h by curves $\gamma(x) : (-\infty, \infty) \rightarrow \Sigma$ which lie in U for $|x|$ sufficiently large. Denote the critical points of h by p_1, \dots, p_N .

Let $s_i \in S$ be the unique point with $\pi(s_i) = p_i$. Then we also assume that as subspaces of $T(T^*\Sigma)$ we have $T_{s_i}S = T(\pi^{-1}(p_i))^{\perp \Omega_\epsilon}$, the symplectic complement to the tangent space of the fiber.

Recall that for r sufficiently large $\omega|_{T^r\Sigma} = d\beta$ where $\beta = e^r \tilde{\lambda}_0 + \epsilon g(r) \tilde{\alpha}$ is a contact form for ϵ sufficiently small. We observe that $\pi^{-1}(\gamma(\mathbb{R})) \cap T^r\Sigma$ is a cylinder C_γ foliated by the circles $F_x = \pi^{-1}(\gamma(x))$. Now, $\tilde{\lambda}_0$ vanishes on the F_x while $\tilde{\alpha}$ does not. Therefore $\ker \beta|_{C_\gamma}$ is a nonsingular line field l transverse to all F_x . In particular l has no closed orbits.

We claim that there exists an almost complex structure J_0 on $T^*\Sigma$ which is tamed by ω and satisfies the following properties. The surfaces $S \cap \pi^{-1}(U)$ and Σ are J_0 holomorphic; the contact planes $\ker \beta$ in $T^r\Sigma$ are J_0 holomorphic for some r sufficiently large; for all critical points p_i the disk $D_i = \pi^{-1}(p_i)$ is J_0 -holomorphic.

The only requirement here which is not well known is the claim that it is possible to find a J_0 along $T^r\Sigma$ which simultaneously makes both the subbundles $\ker \beta$ and $\pi^{-1}(p_i)$ into J_0 -holomorphic distributions. But the

existence of such J_0 is established in a more general context by Theorem 7.4 in the article [4] of J. Coffey.

Let J_t , $0 \leq t \leq 1$ be a family of almost-complex structures on $T^*\Sigma$ coinciding with J_0 outside some $T^s\Sigma$, where $s < r$, and on $\pi^{-1}(U)$, such that S is J_1 holomorphic.

We next claim that for all t the cylinders C_γ can be foliated by circles which bound J_t holomorphic disks. These circles are transverse to l and at the ends of the cylinders the holomorphic disks converge to the perturbed fibers $\pi^{-1}(p_i)$ for p_i a critical point. The union of all disks over all cylinders gives a foliation of $T^{\leq r}\Sigma$ by disks in the relative homotopy class of the fibers.

This claim follows from the theory of filling by holomorphic disks, see [5]. For each γ , the cylinder C_γ is foliated by the boundaries of embedded holomorphic disks near its ends. But as the cylinders are totally real the foliation extends to cover the whole cylinder. The only obstruction in this case is bubbling of holomorphic spheres inside $T^*\Sigma$ and bubbling of disks on the boundary. But as $\pi_2(\Sigma)$ is trivial such spheres do not exist. Bubbling of disks can be excluded as in [5] since all holomorphic disks with boundary on $T^r\Sigma$ must have boundary transverse to l . For embedded boundaries this fixes the homology class and prevents degeneracies.

The disks D_i constructed above are J_t holomorphic for all t and their intersection with S and Σ is transversal and in a single point. Therefore by positivity of intersections the same is true for all intersections of J_0 holomorphic disks with Σ and all J_1 holomorphic disks with S .

We fix a Riemannian metric on $T^*\Sigma$ which decays rapidly along the fibers. Then with respect to the restricted metric the centers of mass of our holomorphic disks give a smooth family of surfaces G_t . By the previous remark, it is clear that G_0 is smoothly isotopic to Σ and G_1 is isotopic to S . \square

Lemma 2.3. *The Lagrangian L can be isotoped to an ω symplectic surface in $T^*\Sigma$ intersecting the fiber over a point p transversally in a single point.*

This will be established using the theory of finite energy planes. This theory was developed in the series of papers [15], [16], [17], [18]. Based on these, work of the first author [10], [11] deduced the existence of finite energy planes in T^*S^2 lying in certain relative homotopy classes. The result we need is the following.

Theorem 2.4. *There exists a Morse-Bott type contact form on the unit cotangent bundle T^1S^2 with an isolated Reeb orbit γ having minimal action*

and Conley-Zehnder index 1. The form can be chosen arbitrarily close to the standard form (where all orbits have action 2π).

The orbit γ is the asymptotic limit of exactly two finite energy planes in T^*S^2 which have intersection number ± 1 with the zero-section.

Here we think of T^*S^2 as an almost-complex symplectic manifold with convex end corresponding to the positive symplectization of T^1S^2 .

Outline of proof of Theorem 2.4

Our contact form will be a perturbation of the standard contact form on T^1S^2 defined with the round metric. With respect to the standard form the space of closed Reeb orbits can be identified with a 2-sphere. We now perturb the contact form following [2] such that the resulting form is of Morse-Bott type. We do this using a function on the orbit space of closed Reeb orbits. We choose a function which is rotationally symmetric and has critical points at two isolated critical points and at a finite number of radii. The critical points on the orbit space correspond to closed Reeb orbits for the perturbed contact form, so we obtain two isolated Reeb orbits and a number of 1-parameter families. Let γ be an isolated Reeb orbit corresponding to a minimum of our function which therefore is of minimal period as required.

We remark here that equivalently our contact form can be realized as the restriction of the Liouville form on T^*S^2 to a hypersurface which is a suitable perturbation of the unit cotangent bundle for the round metric.

To produce finite energy planes asymptotic to the orbit of minimal length we follow the analysis in [10], see also [11]. Our contact manifold T^1S^2 is doubly covered by the contact S^3 . Each of the closed orbits is doubly covered by a closed orbit of the corresponding flow on S^3 . Now the method of filling by holomorphic disks can be applied as in [15] to produce a finite energy plane in the symplectization $\mathbb{R} \times S^3$ asymptotic to a given periodic orbit of minimal length. Such a finite energy plane projects to give a finite energy plane in T^*S^2 asymptotic to twice the corresponding simple orbit.

In T^*S^2 the (unparameterized) finite energy planes asymptotic to twice a simple orbit appear in a 2-dimensional family which foliates a region of T^*S^2 . Now the considerations in [10] imply that this family cannot be compact and a certain subsequence of such planes will bubble to produce a pair of planes asymptotic to the simple orbit. The planes have intersection ± 1 with the zero-section as required.

There exists a circle of isometries of the round metric on S^2 which preserve the geodesic corresponding to γ (simply the rotations about a perpendicular

axis). The isometries lift to a family of symplectomorphisms of T^*S^2 restricting to a family of contactomorphisms of T^1S^2 preserving the Reeb orbit γ . We may assume that this family of symplectomorphisms also preserves the perturbed T^1S^2 and the closed orbit γ . If the almost-complex structure we use is also invariant under this family of rotations then our planes asymptotic to γ must also be invariant. Their uniqueness follows as in [12]. \square

Remark At least if the family is also of minimal length then the same construction can be used to produce a finite energy plane asymptotic to a closed Reeb orbit in one of the 1-parameter families surrounding γ in T^1S^2 . Acting by the circle of isometries we obtain finite energy planes asymptotic to each orbit in this family. The finite energy plane asymptotic to γ has deformation index 0. Further, arguing similarly to Theorem 4.3 in [17], it can be shown that the corresponding Cauchy-Riemann operator is surjective. It seems to be a subtle question to decide whether the finite energy planes in the 1-parameter family are also regular, but we do not need this fact in what follows.

We can think of choosing a contact form on T^1S^2 as being equivalent to choosing a Finsler metric on S^2 , the Reeb orbits then correspond to Finsler geodesics. It is also equivalent to choosing a length function in $T^*\Sigma$ and restricting the Liouville form to the level sets.

Now, in the above construction, the projections to the zero-section of the two planes asymptotic to γ can be compactified to maps from a closed disk with boundary on the corresponding (Finsler) geodesic. For almost-complex structures sufficiently close to the standard one the images of these disks should occupy opposite hemispheres in the S^2 . In any case, if we restrict attention to one relative homotopy class, say that with planes having intersection $+1$ with the zero-section, then we may assume that the projection to S^2 of the plane asymptotic our minimal orbit is disjoint from a segment σ_0 of a geodesic σ which intersects the geodesic corresponding to γ orthogonally at two points.

We may now assume that there exists an $\epsilon > 0$ such that γ and the geodesics in the surrounding 1-parameter family have length less than $2\pi - \epsilon$ and all other closed geodesics have length at least $2\pi + \epsilon$. In particular σ_0 can be taken to have length greater than π . We can further arrange for σ_0 to be disjoint from the 1-parameter family.

Proof of Lemma 2.3 We choose a Finsler metric on Σ and fix an embedded geodesic segment σ_1 . Scaling the metric appropriately we can increase

the length of this segment to π and assume that any closed geodesics on Σ or geodesic segments which start and end on σ_1 have length at least 2π .

On our S^2 from Theorem 2.4 we choose neighborhoods U of σ_0 and V of σ such that the following conditions hold. Geodesic segments starting and ending in U either lie entirely in V or have length at least 2π ; closed noncontractible curves in V have length at least 2π ; closed noncontractible curves in $V \setminus \sigma_0$ have length at least 2π .

We now perform a connected sum of the Finsler surfaces Σ and the S^2 to obtain a new Finsler metric on a surface diffeomorphic to Σ as follows. We do this by removing very small neighborhoods D_0 and D_1 of σ_0 and σ_1 respectively and replacing these with a cylinder. We can extend the Finsler metric over the cylinder so that curves traversing its longitude must have length at least 6π and noncontractible loops in the cylinder have length at least 2π . Furthermore, we may assume that the union of $S^2 \setminus D_0$ and one half of the cylinder glued onto it can be isometrically embedded in $S^2 \setminus \sigma_0$ with a new complete Finsler metric, the embedding being the identity on $S^2 \setminus D_0$. This complete Finsler metric evaluated on tangent vectors is everywhere bounded below by the original metric on the S^2 .

We claim that γ is the closed geodesic of minimal length on our new surface. This follows from the construction for geodesics disjoint from the glued cylinder and for geodesics intersecting both the original S^2 and the original Σ . It is easily guaranteed that the cylinder has a longitudinal foliation by geodesic segments and so geodesics lying entirely in the cylinder must be noncontractible and so also have length at least 2π . Suppose then that a geodesic of length less than 2π lies partly in the cylinder and partly, say, in the original S^2 . Then it can be identified with a geodesic in $S^2 \setminus \sigma_0$ with its complete Finsler metric. Thus if it stays in the region V it must again have length at least 2π , but if it leaves V then it contains a geodesic segment starting and ending in U which also must have length at least 2π . This justifies our claim.

We now vary the symplectic form and tame almost-complex structure inside a compact set so that eventually the form is equal to Ω_ϵ near the zero-section and a symplectic surface S isotopic to L is holomorphic.

Now, for a suitable almost-complex structure J_0 the finite energy plane from Theorem 2.4 asymptotic to γ and having intersection number $+1$ with the S^2 also exists in the cotangent bundle of the new surface Σ . Indeed, it lies disjoint from the region where the connected sum was performed. Then by positivity of intersection with the 1-parameter family of planes asymptotic

to the orbits surrounding γ we see that any finite energy plane asymptotic to γ in this homotopy class must lie in the part of $T^*\Sigma$ projecting to the S^2 and so from Theorem 2.4 the finite energy plane is unique.

We claim that this finite energy plane asymptotic to our Reeb orbit γ continues to exist as the complex structure is deformed to a new structure J_1 . By positivity of intersections we will then see that eventually the plane must intersect S transversally in a single point.

But since γ has minimal period the space of planes asymptotic to γ is compact modulo reparameterizations, see [3]. Combining this with the Fredholm theory for finite energy planes, the space of planes holomorphic with respect to J_1 is cobordant to the space of planes holomorphic with respect to J_0 . There is a unique (regular) J_0 -holomorphic plane and therefore there is an odd number of J_1 -holomorphic planes asymptotic to γ , which justifies our claim.

Hence our finite energy planes persist as the almost-complex structure is deformed and we find an isotopy from the initial plane to a plane which intersects S in a single point. If ϵ is chosen sufficiently small then the planes can both be assumed to be symplectic with respect to ω and the isotopy through ω -symplectic planes. The forms ω on $T^*\Sigma$ were described at the start of this section, we recall that restricted to a level $T^r\Sigma$ we have $\omega = d\beta$ where β is a small perturbation of the Liouville form.

We recall the initial plane may be assumed to intersect the zero-section transversally in a single point and actually each level $T^r\Sigma$ in a curve transverse to the contact structure (see for instance [12]). Using this we can construct an ω -symplectic isotopy taking the initial plane into a fiber through planes similarly intersecting the levels in transverse curves, as follows.

First we remark that given a family $\{\gamma_t\} \subset (T^{r_0}\Sigma, \beta)$ of transverse curves for $t \geq 0$ the union $\{(t, \gamma_t) | t \geq 0\}$ is a symplectic surface in $((0, \infty) \times T^{r_0}\Sigma, d(e^t\beta))$ provided γ_t changes sufficiently slowly with t . (Its tangent space is spanned at a point p by the tangent to γ_t in $\{t\} \times T^{r_0}\Sigma$ and $\frac{\partial}{\partial t} + \frac{\partial\gamma_t}{\partial t}(p)$.) Moreover if $\{\gamma_t\}$ already generates a symplectic surface and we reparameterize the family $\{\gamma_t\}$ to reduce this rate of change then the surface remains symplectic.

To construct our isotopy, given these remarks, it suffices to find an isotopy of transverse curves in some $(T^r\Sigma, \beta)$ from the Reeb orbit corresponding to γ to a fiber of π . Such an isotopy can be constructed as follows. We note that the natural lift (via the metric dual of its derivative) of any embedded curve in Σ to $T^r\Sigma$ is a transverse curve with respect to the (Liouville) contact

structure induced from the metric on Σ . The Reeb orbit corresponding to γ is the natural lift of the meridian and as the meridian contracts to a point we get an isotopy of transverse curves to a curve C^1 close to a fiber. This is also a transverse isotopy with respect to β for ϵ sufficiently small. A transverse curve sufficiently close to a fiber can be moved into a fiber through curves transverse with respect to β . We then construct a symplectic isotopy from the isotopy of transverse curves by observing again that a union of transverse curves γ_s in the contact manifolds $T^*\Sigma$ is a symplectic surface provided that γ_s changes sufficiently slowly with s . With respect to ω the fibers of $T^*\Sigma$ have infinite area and so this isotopy can be arranged to leave a neighborhood of the zero-section inside a given compact set.

Putting everything together, we obtain a proper isotopy from a fiber to a plane intersecting S transversally in a single point. The preimage of S is a surface intersecting a fiber in a single point as required. \square

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